Contests with Expectation-Based Loss-Averse Players

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Job Market Paper

October 25, 2013

Abstract

This paper studies a multiple prize contest with expectation-based loss-averse contestants à la Kőszegi and Rabin (2006). The contest designer first decides on the prize structure - the number and the level of prizes - and contestants then simultaneously undertake costly efforts. Each contestant has private information about his ability, affecting his cost-of-effort. The model predicts that high-ability contestants overexert effort while low-ability contestants exert very little or no effort in comparison to predictions with standard preferences. This result is consistent with the behavior observed in recent laboratory experiments. I also show that the optimal prize allocation in contests differs markedly in the presence of expectation-based loss aversion. In particular, I show that multiple prizes can be optimal when the cost-of-effort function is linear or concave, where standard preferences predict the optimality of a single prize in these cases. Several unequal prizes might be optimal when the cost-of-effort function is convex.

Keywords: contests, all-pay auctions, loss aversion, reference dependence

JEL Classifications: D03, D44, D81, D82

*I am grateful to Wieland Müller, Charles Noussair, Sigrid Suetens, Jan Boone, Jan Potters, Eric van Damme, Christoph Schottmüller, Stefan Trautmann, Patricio Dalton and Botond Kőszegi for their valuable comments. I also thank seminar and conference participants at Tilburg University and Thurgau Experimental Economics Meeting on Reference Dependent Preferences in Kreuzlingen 2013.
1. Introduction

A contest is an event in which participants compete with each other by means of undertaking costly efforts in order to win prizes. There are many economic and social environments that could be described as contests. In sports contests, athletes compete with each other for gold, silver and bronze medals, and in firms employees exert effort in order to be promoted to certain positions, etc. In these examples, the designer’s motive in choosing the prize structure is to induce a general increase of performance, for example, to thrill the audience in sports contests or to obtain the highest output in firms. Since such competitive environments are prevalent in many contexts, contests and their design are studied extensively in the economic literature both theoretically and experimentally.

An important common finding in several experimental studies is that there is a discrepancy between behavior predicted by theory and behavior observed in the lab. In particular, high-ability subjects spend more effort while low-ability subjects spend less or no effort in comparison to predictions with standard preferences (e.g. Barut and Noussair 2002, Noussair and Silver 2006, Ernst and Thoni 2009, Müller and Schotter 2010, Klose and Sheremeta 2012). Some of these studies suggest that this discrepancy may be caused by loss aversion on the part of subjects. One prominent model of loss aversion is K˝ oszegi and Rabin’s model of reference dependent preferences, built on the idea that gain-loss utility is derived from the standard consumption utility and the reference point that is determined by expectations.

Recent empirical studies provide evidence for expectations being determinants of agents’ reference levels. [Post et al. (2008)] examine the behavior of contestants in the TV show “Deal or No Deal”. They find that contestants’ choices can be explained largely by their experience in previous outcomes. Their result suggests that lagged expectations serve as a reference level for contestants in the way predicted by expectation-based loss aversion models. [Abeler et al. (2011)] conduct a real-effort experiment to test whether expectations influence effort provision by manipulating
the rational expectations of subjects. They find that subjects with high expectations work harder relative to subjects with low expectations, consistent with the predictions of expectation-based loss aversion models.

In this paper, I study a multiple-prize contest, generalizing Moldovanu and Sela (2001) contest model, by allowing for expectation-based loss aversion à la Kőszegi and Rabin (2006) on the part of contestants. My model predicts that high-ability contestants exert more effort, while low-ability contestants exert very little or no effort relative to the predictions with standard preferences. This result is consistent with the behavior observed in recent laboratory experiments. The effort provision of the contestants has important implications for the prize structure decided by the contest designer. In fact, I show that the optimal allocation of prizes in a contest changes markedly when contestants are expectation-based loss-averse. In particular, multiple prizes can be optimal when the cost-of-effort function is either linear or concave, whereas standard preferences predict the optimality of a single prize.

Moldovanu and Sela (2001) (henceforth M-S) consider the following contest model. The contest designer first determines the allocation of prizes (the number and the level of prizes) where the total prize sum is fixed. The goal of the contest designer is to maximize the total expected effort of the contestants. Given the prize structure, contestants with standard preferences then choose their effort levels in order to maximize their expected utility. The contestant with the highest effort wins the first prize, the contestant with the second highest effort wins the second prize, and so on until all prizes are distributed. Each contestant bears the cost of effort regardless of winning a prize or not. The cost-of-effort function depends on the ability parameter - which is private information - as well as the effort level. Into this model, I introduce expectation-based loss aversion on the part of contestants in the sense of Kőszegi and Rabin (2006) (henceforth K-R). Following K-R, each contestant, next to the standard consumption utility, derives a gain-loss utility by comparing the actual outcome with his expectations. Particularly, each
contestant compares the actual outcome with all other possible outcomes that could have occurred, and weighs each of these comparisons with the *ex-ante* probability of the alternative outcome occurring. Incorporating expectations as the agent’s reference level induces a bifurcating force among the efforts of high- and low-ability contestants. Intuitively, a high-ability contestant having an *ex-ante* high chances of winning a prize holds high expectations regarding winning a prize. He increases his effort level to further increase his chances of winning in order to avoid the loss sensation associated with not winning a prize. A low-ability contestant having an *ex-ante* low chances of winning a prize holds low expectations regarding winning a prize. He decreases his effort level to further decrease his expectations in order to avoid the feeling of losing a prize. Moreover, if a low-ability contestant is sufficiently loss-averse, the gain-loss utility might dominate the consumption utility. In this case, a contestant exerting positive effort might end up with a negative expected utility. In order to avoid this, he reduces his effort level to the minimum possible level and exerts zero effort.\(^1\)

The contest designer, anticipating the contestants’ behavior, aims to maximize the total expected effort of the contestants. Thus, any change in the contestants’ effort provision has important implications for the designer’s decision about the prize allocation. I show that, in the presence of expectation-based loss aversion, multiple prizes can be optimal when cost-of-effort functions are linear or concave, whereas with standard preferences a single prize is optimal in these cases. Intuitively, if a single prize is announced by the designer, low-ability contestants lose the slim hope of winning a prize and exert very little or no effort. A high-ability contestant, on the other hand, exerts effort aggressively in order to avoid the outcome of not winning a prize, given his high expectations regarding winning a prize. In general, the effort decrease on the part of low-ability contestants dominates the effort increase on the part of high-ability contestants. This results in an overall decrease of the total effort.

\(^1\)The intuition presented here is in line with the “loss contemplation” reasoning for overbidding in auctions presented in Delgado et al. (2008).
expected effort. In order to compensate for the decrease in total expected effort, the contest designer motivates the low-ability contestants by introducing a second, or possibly a third or more prizes. This result is consistent with the experimental findings of Freeman and Gelber (2009). They examine the effort provision in a real-effort tournament in which the prize structure varies. They find that the number of solved mazes is higher when there is multiple differentiated prizes and is lower when there is a single prize.\footnote{\textsuperscript{2}M-S proves that multiple differentiated prizes might be optimal when the cost-of-effort is convex. Freeman and Gelber (2009) use solving mazes as a measure of effort provision in their experiment. In maze solving cost-of-effort is likely to be concave rather than convex since it becomes less costly as you solve more and more mazes due to learning.}

My paper fits well into a recent and growing literature utilizing expectation-based loss aversion in different settings to give a rationale for a variety of empirical findings. Crawford and Meng (2011) analyze field data on cab drivers’ working hours and propose a model of labor supply for cab drivers incorporating the K-R model. Their estimates suggest that their reference-dependent model of labor supply rationalizes the cab drivers’ behavior observed in the field data. Herweg et al. (2010) study the principal agent model with moral hazard by incorporating expectation-based loss aversion. They show that the optimal contract is a binary payment scheme consistent with the observed prevalence of simple contracts. Lange and Ratan (2010) study the first- and second-price sealed bid auctions for a single item with expectation-based loss-averse participants. Their model predicts overbidding in first-prize induced-value auctions, in line with the evidence from laboratory experiments.

In the remainder of this paper, I focus on the two-prize case for ease of exposition. I present the general results for equilibrium effort functions and the optimal prize allocation when there are \( p > 2 \) prizes in the Appendix. In Section 2 I present the model and in Section 3 I introduce further notation and discuss participation in the contest. In Section 4 I focus on the linear cost-of-effort function and derive the equilibrium effort function of the contestants. Afterwards, I state the contest designer’s problem and characterize the optimal prize allocation. I discuss the convex
and concave cost-of-effort function cases in Section 5. I derive the optimal effort function of the contestants and provide a sufficient condition for the optimality of multiple prizes. Section 6 concludes. The proofs are deferred to the Appendix.

2. The Model

Consider a contest with \( p \) prizes with \( V_1 \geq V_2 \geq ... \geq V_p \geq 0 \). Denote the value of the \( j \)-th prize with \( V_j \). The values of the prizes are announced by the contest designer and are common knowledge. The prizes are normalized, so that \( \sum_{i=1}^{p} V_i = 1 \).

Furthermore, let there be \( k \) contestants, with \( k \geq p \). Each contestant has an ability (cost) parameter \( c_i \), which is private information. Ability parameters are drawn independently from a continuous distribution function \( F \) on the interval \([m, 1]\). The distribution function \( F \) is assumed to have a strictly positive and continuous density \( F' > 0 \) and to be common knowledge.

All contestants simultaneously undertake costly efforts. Denote contestant \( i \)'s effort by \( x_i \). Contestant \( i \), exerting effort \( x_i \), bears the cost-of-effort denoted by \( c_i \gamma(x_i) \), where \( \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \) is assumed to be a strictly increasing function with \( \gamma(0) = 0 \). Note that a high \( c_i \) means low ability (higher cost) for contestant \( i \). In the remainder of the text, the contestants having higher \( c_i \)s will be referred to as low-ability contestants and those with low \( c_i \)s will be referred as high-ability contestants. In order to avoid infinite efforts caused by zero costs, the highest possible ability \( m \) is assumed to be strictly positive.

The contestants are assumed to be expectation-based loss-averse in the sense of K-R. I will briefly introduce expectation-based loss aversion and explain how it translates into my model. According to K-R, the overall utility of an agent from consuming the \( n \) dimensional bundle \( \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n \) when having the reference level \( \mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n \) is assumed to have two components: a consumption utility and a gain-loss utility. The consumption utility in a dimension is the standard
outcome-based consumption utility and does not depend on a reference level. The gain-loss utility in a dimension captures how the agent feels about gaining and losing in that dimension, and depends on how consumption in that dimension compares to the reference level. In particular, the overall utility of an agent from consuming \( a = (a_1, \ldots, a_n) \) when having the reference level \( r = (r_1, \ldots, r_n) \) is given by:

\[
v(a|r) = \sum_{l=1}^{n} v_l(a_l) + \sum_{l=1}^{n} \mu(v_l(a_l) - v_l(r_l))
\]

Here, \( v_l \) denotes the consumption utility in dimension \( l \) and \( \mu \) denotes the gain-loss function. The gain-loss function is assumed to satisfy the assumptions put on their value function. In my framework, the consumption space of the contestant has two dimensions, that is \( n = 2 \): that is the prize dimension, i.e. \( a_1 = V_j \) and the effort dimension, i.e. \( a_2 = x_i \). I assume that the consumption utilities in both prize and effort dimensions are given by \( v_j(.) = . \), for \( j \in 1, 2 \). Put verbally, the consumption utility of winning a prize \( V_j \) is identical to the value of that prize. Similarly, the consumption utility of exerting effort \( x_i \) is equal to the cost-of-effort \( c_i \gamma(x_i) \). To discuss the gain-loss utility, it is first necessary to define the “gain-loss function” \( \mu \).

\[
\mu(m) = \begin{cases} 
\eta m, & \text{if } m \geq 0 \\
\eta \lambda m, & \text{if } m < 0,
\end{cases}
\]

where \( \lambda \geq 1 \) is the weight attached to losses relative to gains and \( \eta > 0 \) is the weight attached to gain-loss utility. With this formulation, I assume a constant marginal utility from gains and a larger — in magnitude — marginal disutility from losses. In other words, losses loom larger than gains, however \( \mu(m) \) is not S-shaped, in order
According to K-R, the gain-loss utility is derived from the standard consumption utility and the reference level, as given in equation (1). The reference level is determined endogenously by the environment. I use personal equilibrium as the solution concept. Personal equilibrium states that the decision-maker must choose a state-contingent plan that is optimal given the preferences induced by the plan. That is, expectations should be consistent with optimal behavior given expectations. Given an outcome, the gain-loss utility is derived by comparing the given outcome to all possible outcomes that could have occurred and weighting each comparison with the \textit{ex-ante} probability of the alternative outcome occurring. The gain-loss utility for the given outcome is obtained by summing all these weighted comparisons. The utility from a given outcome is then the sum of the standard consumption utility and the gain-loss utility. The expected utility of a contestant is the weighted average of all possible outcomes, given that the actual outcome itself uncertain.

More precisely, suppose that there are two prizes to be awarded, \( V_1 \geq V_2 \geq 0 \), and \( k > 2 \) contestants. There are three possible outcomes for the contestant in this case: \( (i) \) winning first prize \( V_1 \), \( (ii) \) winning second prize \( V_2 \) and \( (iii) \) not winning any prize. Denote the probabilities with which these outcomes occur by \( p_1, p_2 \) and \( 1 - p_1 - p_2 \), respectively. The outcome that contestant \( i \) wins first prize \( V_1 \) is evaluated as follows:

\[
V_1 + \eta \left\{ p_2 (V_1 - V_2) + (1 - p_1 - p_2) V_1 \right\} + \left\{ -c_i \gamma(x_i) \right\} + 0
\]

\begin{align*}
\text{consumption utility} & \quad \text{gain-loss utility} \\
\text{prize dimension} & \quad \text{consumption utility} \quad \text{gain-loss utility} \\
\text{effort dimension} & \quad 
\end{align*}

(2)

In this formulation, the first term is the consumption utility in the prize dimension.

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\(^3\)I hypothesize that introducing the diminishing marginal sensitivity would sharpen the results presented in this paper.
sion, that is, the consumption utility from winning first prize, which is equal to the value $V_1$. The second term is the gain-loss utility in the prize dimension, which gives the contestant’s feeling of gain or loss from winning first prize $V_1$. This term is obtained by comparing the given outcome - winning first prize - to all possible outcomes, namely winning second prize or not winning anything. Compared to the alternative outcome that the contestant ends up with second prize $V_2$, which happens with a probability $p_2$, he experiences a gain of $V_1 - V_2$; meanwhile, compared to the alternative outcome where the contestant ends up not winning any prize, which happens with a probability $(1 - p_1 - p_2)$, he experiences a gain of $V_1$. The coefficient $\eta$ is the weight of the gain-loss utility, which measures the weight attached to the gain-loss utility relative to the consumption utility. Note that in all these comparisons the contestant is in the gain domain, since winning first prize is the best outcome. The last term in (2) is the consumption utility in the effort dimension, namely the standard disutility of losing effort $x_i$. The gain-loss utility in the effort dimension is simply zero, since the expected and the actual effort choices of the contestant coincide.

Similarly, the utility of contestant $i$ from winning second prize $V_2$ is formulated as follows:

$$
V_2 + \eta \left\{ p_1 \lambda (V_2 - V_1) + (1 - p_1 - p_2)V_2 \right\} + \frac{-c_i \gamma(x_i)}{\text{consumption utility}} + \frac{0}{\text{gain-loss utility}}.
$$

(3)

In the above evaluation, different from the first one, the loss aversion index $\lambda$ comes into the picture. This is because the contestant is in the loss domain when he compares winning second prize $V_2$ to the alternative outcome of winning first prize $V_1$. 

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The utility of contestant \(i\) from not winning any prize is evaluated in the same way:

\[
EU = \begin{align*}
\text{consumption utility} & + \eta \left\{ p_1 \lambda (-V_1) + p_2 \lambda (-V_2) \right\} + c_i \gamma(x_i) \\
\text{gain-loss utility} & + 0 \\
\text{prize dimension} & + \\
\text{consumption utility} & + 0 \\
\text{gain-loss utility} & + \\
\text{effort dimension} & .
\end{align*} \quad (4)
\]

In comparisons of not winning any prize to the alternative outcomes of winning first and second prize, the contestant is in the loss domain. Note that not winning any prize is the least favorable outcome for the contestant, since each contestant bears the cost-of-effort regardless of winning a prize.

The actual outcome being uncertain, the expected utility of contestant \(i\) with type \(c_i\) is then given by the sum of (2), (3) and (4):

\[
EU = p_1 \{ V_1 + \eta (p_2 (V_1 - V_2) + (1 - p_1 - p_2) V_1) - c_i \gamma(x_i) \} \\
+ p_2 \{ V_2 + \eta (p_1 \lambda (V_2 - V_1) + (1 - p_1 - p_2) V_2) - c_i \gamma(x_i) \} \\
+ (1 - p_1 - p_2) \{ \eta (p_1 \lambda (-V_1) + p_2 \lambda (-V_2)) c_i \gamma(x_i) \}. \quad (5)
\]

Note that the probabilities \(p_1\), \(p_2\) and \((1 - p_1 - p_2)\) are affected by the effort that the contestant exerts: \(p_1\) is the probability that contestant \(i\) meets \((k - 1)\) competitors and that all of them exert less effort than him, and \(p_2\) is the probability that he meets \((k - 1)\) competitors and that \((k - 2)\) of them exert less effort than him while one competitor exerts more effort. The probability of not winning any prize is given by \((1 - p_1 - p_2)\). Note that by changing his effort level, each contestant affects the probability of winning a prize as well the endogenous reference point. Letting \(\lambda = 1\) and \(\eta = 1\) equation (5) reduces to the expected utility under standard preferences as formulated in M-S.

The timing of the contest game is as follows. In the first stage, the contest
designer chooses the number and the level of the prizes in order to maximize the total expected effort. The designer’s revenue is the sum of expected efforts. The prize sum is fixed and assumed to be the normalized $\sum_{i=1}^{k} V_i$. In the second stage, given the prize structure, the contestants choose their effort levels in order to maximize their expected utility. The contestant with the highest effort wins first prize $V_1$, and the contestant with the second highest effort wins second prize $V_2$. Each contestant bears the cost-of-effort regardless of winning any prize.

3. Participation in the Contest

Before launching into the discussing participation in the contest, it is convenient to introduce the following notation to ease the exposition. First, define $\Lambda = \eta(\lambda - 1)$, where $\eta$ is the weight placed on the gain-loss utility relative to the consumption utility and $\lambda$ is the degree of loss aversion. $\Lambda$ is interpreted as an overall measure of an agent’s degree of loss aversion (see also [Herweg et al. (2010) and Eisenhuth and Ewers (2012)]). $\Lambda > 0$ for the loss-averse agent and $\Lambda = 0$ whenever the agent has standard preferences. Rearranging the terms in equation (5) and substituting $\Lambda = \eta(\lambda - 1)$, the expected utility of contestant $i$ can be rewritten as follows:

$$EU = p_1 V_1 + p_2 V_2 - c_i \gamma(x_i)$$

$$-\Lambda \left\{ p_1 p_2 (V_1 - V_2) + (1 - p_1 - p_2)(p_1 V_1 + p_2 V_2) \right\}.$$ 

Second, let $F_s(c)$, $s \in \{1, 2\}$, denote the probability that a contestant with type $c$ has a higher type than $s - 1$ of his $k - 1$ competitors while he has a lower type than $k - s$ of his $k - 1$ competitors. To illustrate, $F_1(c)$ is the probability that all remaining $(k - 1)$ contestants have higher types, that is they are less able, and $F_2(c)$ is the probability that $(k - 2)$ of the remaining contestants have lower types while one of them has a higher type. In other words, $F_1$ and $F_2$ are the first-order
statistics. Recall that a low-ability contestant has a higher $c$, leading to higher costs. Note that in equilibrium it is assumed that contestant $i$ exerts higher effort than his competitors with higher types. Contestant $i$ affects these probabilities of winning the first and the second prize by choosing his effort level $x$.

Now I will discuss the participation in the contest. Note that when $\Lambda = 0$, the expected utility of the agent in equation (6) equals the expected consumption utility. In this case, the agent has standard preferences but no gain-loss sensation. M-S show that there is full participation in the contest under the assumption of standard preferences, that is when $\Lambda = 0$. Whenever $\Lambda > 0$, the agent has the expected gain-loss utility, next to the expected consumption utility. Given the fact that first prize is always larger than or equal to second prize, the gain-loss utility – the second line of the equation (6) – is either zero or negative. Depending on the relative magnitudes of the gain-loss utility and the standard consumption utility, the agent may end up with negative expected utility. Put differently, the agent has a non-negative expected utility only if the expected gain-loss utility does not dominate the expected consumption utility. If the agent is sufficiently loss-averse, that is when $\Lambda$ is sufficiently large, then he may end up with negative expected utility whenever he exerts positive effort. In order to avoid this situation, he exerts zero effort and stays out of the contest. Intuitively, whenever loss aversion is too pronounced, the primary concern of a contestant with a low probability of winning becomes reducing the likelihood of possible losses. In this case, he gives up the slim hope of winning a prize and avoid losses by reducing his effort level to zero.

Rearranging the terms in the expected utility given by equation (6), I obtain a condition that guarantees that a contestant’s participation in the contest. A contestant with ability parameter $c$ derives a non-negative expected utility from participating in the contest only if:
\[
\frac{F_1(c)^2V_1 + 2F_1(c)F_2(c)V_2 + F_2(c)^2V_2}{F_1(c)V_1 + F_2(c)V_2} > 1 - \frac{1}{\Lambda}.
\] (7)

Note that whenever \( \Lambda \leq 1 \), the condition in (7) is satisfied for any parameter \( c \in [m, 1] \), implying each contestant has a nonnegative expected utility. However, whenever \( \Lambda \geq 1 \), condition (7) may be violated for some contestants with sufficiently small probabilities of winning a prize. In other words, only contestants with a sufficiently large probability of winning a prize exert positive effort, while those with a sufficiently small probability of winning a prize reduces their effort to zero, and do not participate in the contest. Therefore, we obtain:

**Lemma 1.** There is full participation in the contest when \( \Lambda \leq 1 \). When \( \Lambda > 1 \), there is a critical type \( \tilde{c} \) satisfying (7) with equality such that contestants with the ability \( c > \tilde{c} \) drop-out by exerting zero effort.

Lemma 1 guarantees full participation in the contest whenever \( \Lambda \leq 1 \) (see also Herweg et al. (2010), Eisenhuth and Ewers (2012)). To put differently, when players are sufficiently loss averse, \( \Lambda > 1 \), there is a group of players who exerts zero effort and do not participate in the contest. This result is consistent with the recent experimental evidence (see Müller and Schotter (2010), Barut and Noussair (2002), Noussair and Silver (2006), Klose and Sheremeta (2012), Ernst and Thöni (2009)).

4. Linear Cost Functions

In this section, I will solve the contestant’s and the designer’s problems, respectively, for the linear cost-of-effort function. I will first derive the optimal behavior of the contestants for a given prize structure. Next, given the optimal behavior of the contestants for any prize structure, I will characterize the optimal prize allocation.
4.1. Contestants’ Problem

Assume that the contestants have linear cost-of-effort functions, that is $\gamma(x) = x$. The following proposition displays the equilibrium effort function of a contestant when there are two prizes to be awarded and $k > 2$ loss-averse contestants.

**Proposition 1** Assume that there are two prizes $V_1 \geq V_2 \geq 0$ to be awarded and $k > 2$ contestants. If $\Lambda > 1$, then there exists a critical type $\tilde{c}$ satisfying (7) with equality, such that in equilibrium contestants with $c \geq \tilde{c}$ exert zero effort and contestants with $c < \tilde{c}$ exert effort according to:

$$b(c) = A(c)V_1 + B(c)V_2$$

where the coefficients of the first and second prize are given by:

$$A(c) = (1 - \Lambda) \int_{\tilde{c}}^{\tilde{c}} -\frac{1}{a} F_1'(a) \, da + \Lambda \int_{\tilde{c}}^{\tilde{c}} -\frac{1}{a} (F_1^2(a))' \, da$$

and

$$B(c) = (1 - \Lambda) \int_{\tilde{c}}^{\tilde{c}} -\frac{1}{a} F_2'(a) \, da + \Lambda \int_{\tilde{c}}^{\tilde{c}} -\frac{1}{a} ((F_2^2(a))' + (2F_1(a)F_2(a))') \, da.$$

If $\Lambda \leq 1$, then each contestant exerts effort according to equation (8), where $A(c)$ and $B(c)$ are as in equations (9) and (10) with $\tilde{c} = 1$.

**Proof.** See appendix A.

The equilibrium effort function for the general case with $p$ prizes and $k \geq p$ contestants is derived in Appendix C.

By Lemma 1, full participation in the contest is guaranteed when $\Lambda \leq 1$. In equilibrium, each contestant exerts an effort equal to a weighted sum of first and second prize. The weights of the prizes differ for each contestant depending on his chances of winning first and second prize. When $\Lambda > 1$, there is a subset of
contestants who exert 0 effort in equilibrium. Note that by letting $\Lambda = 0$, the above equilibrium effort functions reduce to those with standard preferences, formulated in M-S.

The following example illustrates the equilibrium effort function of contestants under a uniform distribution of abilities.

**Example 1** Assume that there are $k = 3$ contestants whose abilities are drawn from the uniform distribution $F(c) = 2c - 1$ on the interval $[1/2, 1]$. First, let $\Lambda = 0.8$, guaranteeing that each contestant participates in the contest (see Lemma 1). Figure 1 depicts the equilibrium effort function in the presence of standard preferences and expectation-based reference-dependent preferences.

![Figure 1: Equilibrium Effort Functions](image)

(a) Single prize, $V_1 = 1$ and $V_2 = 0$.  
(b) Two equal prizes, $V_1 = V_2 = 0.5$.

*Notes:* The left panel depicts the equilibrium effort functions when the designer awards a single prize. The right panel depicts the equilibrium effort functions when the designer awards two equal prizes. The degree of loss aversion of the contestants is $\Lambda = 0.8$.

Recall that the expected utility of a contestant (see equation 6) has two parts: the expected consumption utility and the expected gain-loss utility. The expected consumption utility is equal to the expected utility of a contestant with standard preferences. Therefore, the difference in the equilibrium behavior between a contestant with standard preferences and a contestant with reference-dependent preferences stems from the expected gain-loss utility. Recall that the gain-loss utility is
derived by comparing the actual outcome with the contestant’s expectations. This is the point where expectations come into the picture. A contestant with high ability has an *ex-ante* high probability of winning a prize, leading him to have high expectations. In order to avoid the loss of not winning a prize, he increases his probability of winning by increasing his effort level further. On the other hand, a contestant with low ability has an *ex-ante* low probability of winning a prize, leading him to have low expectations. In order to reduce the scope of possible losses, he reduces his expectations further by lowering his effort level. Therefore, expectation-based loss aversion incentivizes high-ability contestants to increase their effort level while it induces low-ability contestants to lower their efforts levels. Therefore, high-ability contestants exert more effort while low-ability ones exert less effort in comparison to the predictions of standard preferences.

Now let $\Lambda = 1.5$, in which case there is a critical type $\tilde{c}$ satisfying condition (7) with equality such that any type $c \geq \tilde{c}$ exerts zero effort by Lemma 1. Figure 2 depicts the equilibrium effort functions when $\Lambda = 1.5$.

**Figure 2: Equilibrium Effort Functions**

![Figure 2](image)

(a) Single prize, $V_1 = 1$ and $V_2 = 0$.  
(b) Two equal prizes, $V_1 = 1 = V_2 = 0.5$.

**Notes:** The left panel depicts the equilibrium effort functions when the designer awards a single prize. The right panel depicts the equilibrium effort functions when the designer awards two equal prizes. The degree of the loss aversion of the contestants is $\Lambda = 1.5$.

When the overall degree of loss aversion $\Lambda$ exceeds 1, we still see the aggressive effort provision of high-ability contestants and the under-exertion of effort of low-
ability contestants. In addition to these findings, the dropping-out behavior of low-ability contestants occurs. Intuitively, when a low-ability contestant is sufficiently loss-averse, the gain-loss utility dominates the standard consumption utility. In this case, the contestant focuses on reducing the net loss arising from the gain-loss utility and exerts zero effort. These results are consistent with the experimental evidence presented in Müller and Schotter (2010).

4.2. Designer’s Problem

Given the optimal behavior of contestants for any prize allocation, the contest designer chooses the number and the level of the prizes. The goal of the contest designer is to maximize his expected revenue, namely the total expected effort exerted by contestants. Let $V_2 = \alpha$ and $V_1 = 1 - \alpha$, where $0 \leq \alpha \leq 1/2$.

Recall that whenever $\Lambda > 1$, there is a positive mass of types $c \geq \tilde{c}$ exerting zero effort by Lemma 1. Contestants with $c < \tilde{c}$ exert effort according to equation (8). When $\Lambda \leq 1$, there is full-participation, so that $\tilde{c} = 1$. The average effort of each contestant is given by:

$$
\int_{\tilde{c}}^{\infty} b(c) F'(c) dc = \int_{\tilde{c}}^{\infty} (1 - \alpha) A(c) + \alpha B(c) F'(c) dc.
$$

(11)

where $A(c)$ and $B(c)$ are given by equations (9) and (10). As there are $k$ contestants, the designer’s problem is given by:

$$
\max_{0 \leq \alpha \leq 1/2} k \int_{\tilde{c}}^{\infty} (A(c) + \alpha (B(c) - A(c))) F'(c) dc.
$$

(12)

Since the maximization is over $\alpha$, the designer’s problem can be written as follows:

$$
\max_{0 \leq \alpha \leq 1/2} \alpha \int_{\tilde{c}}^{\infty} (B(c) - A(c)) F'(c) dc.
$$

(13)
The solution to the designer’s problem depends on the sign of the integral in equation (13): it is optimal to award a single prize if the integral is negative, and to award two equal prizes otherwise. Note that awarding two unequal prizes is never optimal due to the linearity of the program. The sign of the integral depends on the specific properties of the distribution function \( F \) of abilities, the number of contestants \( k \) and the degree of loss aversion \( \Lambda \).

**Proposition 2** Assume that there are at most two prizes to be awarded with \( V_1 \geq V_2 \geq 0 \) and \( k > 2 \) contestants with linear cost-of-effort functions. Then it is optimal to allocate the whole prize sum into a single prize if, and only if:

\[
\int_{m}^{\bar{c}} (B(c) - A(c)) F'(c) dc < 0 \quad (14)
\]

and to award two equal prizes otherwise.

**Proof.** See Appendix B.

The solution to the designer’s problem for the general case with \( p \) prizes, \( k \geq p \) contestants and any \( \Lambda \) is derived in Appendix D. The following example illustrates the optimal prize allocation under a uniform distribution of abilities.

**Example 2** Assume that there are 3 contestants, whose abilities are drawn from a uniform distribution \( F(c) = 2c - 1 \) on the interval \([0.5, 1]\). Figure 3 depicts the equilibrium effort functions when the designer announces a single grand prize, \( b^{(1,0)} \), and two equal prizes, \( b^{(0.5,0.5)} \) separately. The indices \((1,0)\) and \((0.5,0.5)\) refer to the prize allocations \( V_1 = 1, V_2 = 0 \) and \( V_1 = 0.5, V_2 = 0.5 \), respectively. The dashed and the bold lines are the equilibrium effort functions under the assumption of standard preferences and expectation-based reference-dependent preferences, respectively.

In general — for both preference types — a second prize motivates low-ability contestants to increase their effort level. Intuitively, low-ability contestants would give up the competition if there is only a single prize and exert more effort when the
contest designer announces a second prize. On the other hand, a second prize will
give high-ability contestants an incentive to lower their effort levels. This is because
high-ability contestants are mainly competing for first prize and introducing a second
prize will lower the value of the first (since the prize sum is constant). Figure 3 illustrates the effort decrease of high-ability contestants and the effort decrease of
low-ability ones in the presence of a second prize.

The contest designer decides on whether to introduce a second prize by comparing
the differences in effort provision of high and low-ability contestants. If the increase
in the total expected effort by low-ability contestants — in the presence of a second
prize — dominates the decrease in total expected effort by high-ability contestants,
then the contest designer is better off by introducing a second prize.

Figure 3: The Beneficial Effect of Second Prize

Notes: The figure depicts the optimal effort functions in the presence of a single and two
prizes. The dashed lines are the equilibrium effort functions under the assumption of
standard preferences, while the bold ones are under the assumption of expectation-based
reference-dependent preferences. $\Lambda = 0.8$ in the left panel and $\Lambda = 1.5$ in the right panel.

M-S show that when contestants have standard preferences, the effort increase
of low-ability contestants does not compensate for the effort decrease of high-ability
contestants relative to the single prize case, so that a single first prize is optimal.
A reasonable conjecture is that the result of this comparison will depend on the
number of contestants and the specific properties of the ability distribution. M-S
show that their result is independent of the variables under consideration. When contestants have expectation-based reference-dependent preferences, however, the comparisons of effort provision across types depend on the variables.

For the specific values taken in this example, and contrary to the case of standard preferences, it is optimal to award two equal prizes. This is because when contestants are expectation-based loss-averse, the low-ability contestants provide little or no effort and high-ability contestants aggressively exert effort in comparison to the predictions of standard preferences. In this case, the effort increase of low-ability contestants does compensate for the effort decrease of high-ability contestants relative to the single prize case. As such, the contest designer is better off when he allocates the total prize sum as two equal prizes. The optimality of two equal prizes - rather than two unequal prizes - is due to the linearity of the program (see proof of Proposition 2). When $\Lambda = 1.5$, the effort decrease of low-ability contestants becomes more prominent due to drop-outs, depicted in the right panel of Figure 3.

Figure 4 depicts the optimal prize structure for the combination of different values for $k$ and $m$ under a uniform distribution of abilities. For the values in the shaded area it is optimal to award two equal prizes, while a single prize is optimal in the unshaded area. As the overall degree of loss aversion increases, the area over which two equal prizes are optimal expands.

As the number of contestants $k$ increases, keeping everything else constant, the beneficial effect of second prize on the total expected effort increases. Intuitively, a contestant has a lower probability of winning when there are more competitors. All but the high-ability contestants will have lower expectations regarding winning a prize if there are more competitors. The contest designer motivates these contestants by introducing a second prize, leading him to obtain a higher total expected effort.

As the ability parameter of the lowest type $m$ increases, keeping everything else constant, the competitors of a contestant become more able relative to the case of a smaller $m$. In this case, each contestant has lower expectations regarding winning a
Notes: The figure illustrates the optimal allocation of prizes depending on the number of contestants $k$ and the lowest type $m$. For the values of $k$ and $m$ in the unshaded area, it is optimal to award a single prize, while for the values in the shaded area it is optimal to allocate the total prize sum as two equal prizes.

Introducing a second prize increases the expectations of contestants regarding winning a prize, motivating contestants to exert more effort, leading to an increase in the total expected effort.

5. Concave and Convex Cost Functions

In this section, I will solve the contestants’ and the designer’s problem, respectively, for convex or concave cost-of-effort functions, similar to the previous section. I will first derive the optimal behavior of the contestants for a given prize structure. Next, given the optimal behavior of the contestants for any prize structure, I will characterize the optimal prize allocation.

5.1. Contestants’ Problem

Assume that the contestants have either concave or convex cost-of-effort functions with $\gamma(0) = 0$ and $\gamma$ as an increasing function. The following proposition displays the equilibrium effort function of a contestant when there are two prizes to be awarded and $k > 2$ contestants.
**Proposition 3** Assume that there are two prizes $V_1 \geq V_2 \geq 0$ to be awarded and $k > 2$ contestants. If $\Lambda > 1$, then there exists a critical type $\tilde{c}$ satisfying (7) equality such that while in equilibrium contestants with $c \geq \tilde{c}$ exert zero effort and contestants with $c < \tilde{c}$ exert effort according to:

$$b(c) = \gamma^{-1}(A(c)V_1 + B(c)V_2),$$

where the coefficients of first and second prize are given by equations (9) and (10), respectively. If $\Lambda \leq 1$, then the optimal effort for all types is positive and given by equation (15), where $A(c)$ and $B(c)$ are defined by equations (9) and (10) with $\tilde{c} = 1$.

**Proof.** See Appendix A.

The equilibrium effort function for the general case with $p$ prizes and $k \geq p$ contestants is derived in Appendix C. The equilibrium effort of each contestant is given by a simple transformation of the equilibrium effort obtained in the linear cost case. Note that when $\Lambda = 0$, the equilibrium above reduces to that with standard preferences formulated in M-S.

The following example illustrates the equilibrium effort function of contestants with convex and concave cost-of-effort functions, respectively.

**Example 3** Assume that there are $k = 3$ contestants, whose abilities are drawn independently from the uniform distribution $F(c) = 2c - 1$ on the interval $[1/2, 1]$, as in example 1. Assume that the convex cost-of-effort function is $\gamma(x) = x^2$ and the concave cost-of-effort function is $\gamma(x) = \sqrt{x}$. Figure 5 depicts the equilibrium effort functions when contestants have concave cost-of-effort functions. The upper panel (Figures 5a and 5b) illustrates the effort provision in equilibrium when there is full participation in the contest, $\Lambda = 0.8$. The lower panel, (Figures 5c and 5d) illustrates the case where low-ability contestants drop out, namely $\Lambda = 1.5$.

The equilibrium effort functions in the case of convex or concave cost-of-effort
Figure 5: Equilibrium Effort Functions for Concave Costs \( \gamma(x) = \sqrt{x} \)

Notes: The left panels depict the equilibrium effort curves when there is a single prize, while the right panels depict the equilibrium effort curves when there are two equal prizes. The upper and the lower panels illustrate the equilibrium effort curves, respectively, for \( \Lambda = 0.8 \) and \( \Lambda = 1.5 \).

functions is obtained by a simple transformation of the equilibrium effort curve found in the linear cost-of-effort case, so that the intuition provided in Example 1 applies to the cases of concave or convex cost-of-effort functions in the same way. In particular, high-ability contestants aggressively exert effort while low-ability contestants exert little or no effort, relative to the predictions with standard preferences. This is because a contestant with high ability, holding high expectations, exerts effort aggressively in order to avoid the loss of not winning a prize. On the other hand, a contestant with low ability, holding low expectations, exerts little effort to reduce his expectations further in order to minimize the loss sensation stemming from their gain-loss utility. Whenever a contestant is sufficiently loss-averse, \( \Lambda > 1 \), low-ability contestants exert zero effort, dropping out of the contest. This is because, for a
Figure 6: Equilibrium Effort Functions for Convex Costs $\gamma(x) = x^2$

Notes: The left panels depict the equilibrium effort curves when the designer awards a single prize. The right panels depict the equilibrium effort curves when the designer awards two equal prizes. For both structures, the degree of loss aversion of the contestants is $\Lambda = 1.5$.

low-ability contestant, the gain-loss utility might dominate the standard consumption utility. In this case, the contestant’s primary concern becomes avoiding possible losses, incentivizing him to drop his effort level to zero.

5.2. Designer’s Problem

Let $V_2 = \alpha$ and $V_1 = 1 - \alpha$, where $0 \leq \alpha \leq 1/2$. Analogous to the case of linear cost-of-effort functions, the average effort of each contestant with a convex or concave cost-of-effort function is given by:

$$\int_m^{\tilde{c}} \gamma^{-1} \left( A(c) + \alpha (B(c) - A(c)) \right) F'(c) dc \quad (16)$$
where $A(c)$ and $B(c)$ are given by equations (9) and (10). Note that whenever $\Lambda \leq 1$, full participation in the contest is guaranteed (see Lemma 1) so that $\tilde{c} = 1$. Since there are $k$ contestants, the total expected effort — the revenue of the designer — is given by:

$$R(\alpha) = k \int_m^{\tilde{c}} \gamma^{-1}(A(c) + \alpha(B(c) - A(c)))F'(c)dc. \quad (17)$$

Since the goal of the designer is to maximize the total expected effort, the designer’s problem becomes:

$$\max_{0 \leq \alpha \leq 1/2} k \int_m^{\tilde{c}} \gamma^{-1}(A(c) + \alpha(B(c) - A(c)))F'(c)dc. \quad (18)$$

The solution to the designer’s problem depends on the shape of the revenue function $R(\alpha)$. More specifically, awarding a single prize is optimal if $R(\alpha)$ is strictly decreasing, that is if the revenue function has its maximum at $\alpha = 0$. Otherwise, the revenue function $R(\alpha)$ might have its maximum at $\alpha \neq 0$, leading to the optimality of the two prizes. The shape of the revenue function $R(\alpha)$ depends on the degree of loss aversion $\Lambda$ as well as the number of contestants and the specific properties of the distribution function $F$. If the shape of the revenue function $R(\alpha)$ is concave, the maximization problem of the designer might have an interior solution with $\alpha^* \in (0, 1)$. In this case, two unequal prizes become optimal, in contrast to the case of linear cost-of-efforts. In the following proposition, I provide a sufficient condition for the optimality of two prizes.

**Proposition 4** Assume that there are at most two prizes to be awarded with $V_1 \geq V_2 \geq 0$ and $k > 2$ contestants with convex or concave cost-of-effort functions. A sufficient condition for the optimality of two prizes is given by:

$$\int_m^{\tilde{c}} (B(c) - A(c))g'(A(c))F'(c)dc > 0. \quad (19)$$
If condition (19) is satisfied, then it is optimal to award two prizes $V_1 = 1 - \alpha^*$ and $V_2 = \alpha^*$ with $R'(\alpha^*) = 0$, otherwise it is optimal to award a single prize.

**Proof.** See Appendix B.

Letting $\Lambda = 0$, the condition (19) reduces to that provided in M-S. The integral in condition (19) is an increasing function of the number of competitors. Hence, for a given cost function, if the number of competitors is high enough then it is optimal to award two prizes. The ratio of the prizes depends on the distribution of types as well as their degree of loss aversion.

If the cost-of-effort is concave and there is full participation in the contest - that is if $\Lambda \leq 1$ - then the shape of the revenue function $R(\alpha)$ is convex. In this case, the maximization problem in equation (18) has corner solutions. In other words, it is optimal to award either a single prize or two equal prizes, obtaining the following corollary:

**Corollary 1** Assume that there are at most two prizes to be awarded with $V_1 \geq V_2 \geq 0$ and $k > 2$ contestants with concave cost-of-effort functions. If $\Lambda \leq 1$, then it is optimal to award either a single prize or two equal prizes.

**Proof.** See Appendix B.

The following example illustrates the optimal prize allocation for concave cost-of-efforts under a uniform distribution of abilities.

**Example 4** Assume that there are $k$ contestants, whose abilities are drawn from a uniform distribution $F(c) = 2c - 1$ on the interval $[1/2, 1]$. Assume, moreover, that the cost-of-effort function is $\gamma(x) = \sqrt{x}$. Figure 7 depicts the equilibrium effort functions in the case of a single prize, $b^{(1,0)}$, and two equal prizes, $b^{(0.5,0.5)}$. The dashed and the bold lines are the equilibrium effort curves under the assumption of standard preferences and expectation-based reference-dependent preferences, respectively.

Since the equilibrium effort curve in the case of a concave cost-of-effort function is a transformation of that obtained in the case of a linear cost-of-effort function,
the intuition presented in Example 2 applies to this example as well. Particularly, introducing a second prize motivates low-ability contestants to increase their effort levels while leading high-ability contestants to lower their effort levels. Figure 7 illustrates the decrease in effort of high-ability types and the increase in effort of low-ability types, in the presence of a second prize. If the former effect compensates for the latter one, then it is optimal to award a second prize.

Figure 7: The Beneficial Effect of Second Prize

Notes: The figure depicts the optimal effort curves in the presence of a single prize or two prizes. The dashed lines are the equilibrium effort curves under the assumption of standard preferences, while the bold ones are under the assumption of expectation-based reference-dependent preferences. The left panel illustrates the curves for \( \Lambda = 0.8 \) and the right panel for \( \Lambda = 1.5 \).

M-S show that when contestants have standard preferences, it is optimal to award a single prize in the case of concave cost-of-effort functions. As in the case of linear cost-of-effort, they show that this prediction is independent of the number of contestants and the ability distribution. When contestants have expectation-based reference-dependent preferences, however, awarding a second prize can be optimal depending on the number of players and the ability distribution. Figure 8 depicts the optimal prize structure for different values of \( k \) and \( m \) under a uniform distribution of abilities.

Figure 8a illustrates the case when there is full participation in the contest. In this case, it is optimal to award two equal prizes for the values of \( k \) and \( m \) in the
Figure 8: **Optimal Prize Allocation**

Notes: The figure illustrates the optimal allocation of prizes depending on the number of contestants \( k \) and the lowest type \( m \). For the values of \( k \) and \( m \) in the unshaded area, it is optimal to award a single prize. When \( \Lambda = 0.8 \) for the values of \( k \) and \( m \) in the unshaded area, it is optimal to award two equal prizes, and when \( \Lambda = 1.5 \) it might be optimal to allocate the prize sum as two unequal prizes.

shaded area, and to award a single prize in the remaining area. In comparison to the linear cost-of-effort functions, the optimality of a single prize becomes less likely. This is because, with the concave cost-of-effort functions, the ability range over which contestants exert little effort is larger relative to linear cost-of-effort functions. Figure 8b illustrates the case when low-ability contestants drop out. In this case, the beneficial effect of a second prize becomes more prominent for the contest designer, so that the area over which it is optimal to offer a single prize shrinks. Distinct from case of linear cost-of-effort, it can be optimal to award two unequal prizes when there is dropping-out behavior. As the values of \( k \) and \( m \) increase, awarding two prizes becomes optimal, as discussed in Example 2.

6. Conclusion

In this paper, I studied a multiple prize contest under incomplete information, generalized the contest model of Moldovanu and Sela (2001) by allowing for expectation-based loss aversion according to Kőszegi and Rabin (2006). The model presented in
this paper is able to align the common experimental finding that high-ability contestants exert effort aggressively while low-ability contestants exert very little effort or drop out of the contest, in comparison to the predictions with standard preferences.

An expectation-based loss-averse contestant has an expected gain-loss utility next to his expected consumption utility. The expected gain-loss utility measures the net loss sensation derived by comparing the actual outcome to all other alternative outcomes that might have occurred. High-ability and low-ability contestants have different incentives in order to avoid the feeling of loss stemming from the gain-loss utility. Intuitively, a high-ability contestant, having high expectations regarding winning a prize, increases his effort level in order to avoid the loss of not winning a prize. A low-ability contestant, having low expectations regarding winning a prize, decreases his expectations further by exerting very little effort to avoid the situation of losing a prize. When loss aversion is sufficiently pronounced, gain-loss utility dominates the standard consumption utility. In this case, and in order to avoid the net loss, a low-ability contestant exerts zero effort and drops out of the contest.

The second main result is that in the presence of expectation-based loss aversion, awarding multiple prizes can be optimal where standard preferences predict the optimality of a single prize. The beneficial effect of a second prize becomes more prominent when contestants are expectation-based loss-averse. The reason is that low-ability contestants provide little effort due to their low expectations regarding winning a prize. The contest designer can increase his revenue — total expected effort — by motivating low-ability contestants with a second or possibly a third or more prizes. The optimality of multiple prizes is consistent with the prevalence of multiple prize contests in the real world.
Appendices

A. Derivation of Equilibria

Proof of Proposition 1. Assume that all contestants except \( i \) exert effort according to the function \( b \). Moreover assume that \( b \) is strictly monotonic and differentiable. I will derive the optimal effort function first for the case when there is full participation in the contests (when \( \Lambda \leq 1 \)) and then for the case when some contestants drop out the contest (when \( \Lambda > 1 \)).

Suppose that each contestant participates in the contest, that is \( \Lambda \leq 1 \). The maximization problem of the contestant \( i \) is:

\[
\max x \quad \{ p_1 \{ V_1 + \eta(p_2(V_1 - V_2) + (1 - p_1 - p_2)V_1) - cx \} \\
+ p_2 \{ V_2 + \eta(p_1\lambda(V_2 - V_1) + (1 - p_1 - p_2)V_2) - cx \} \} \\
+ (1 - p_1 - p_2)\{ \eta(p_1\lambda(-V_1) + p_2\lambda(-V_2)) - cx \} \} .
\]

(20)

where the probabilities of winning the first and the second prize, \( p_1 \) and \( p_2 \), are defined as

\[
p_1 = (1 - F(b^{-1}(x)))^{k-1}
\]

(21)

\[
p_2 = (k - 1)(1 - F(b^{-1}(x)))^{k-2}F(b^{-1}(x)).
\]

\( p_1 \) is the probability that all remaining \((k - 1)\) contestants have higher types, that is they are less able, and \( p_2 \) is the probability that \((k - 2)\) of the remaining contestants have lower types while one of them has a higher type. Note that a contestant affects these probabilities of winning the first and the second prize by choosing his effort level \( x \).

Denote the inverse effort function \( b^{-1} \) by \( y \). Substituting \( b^{-1} \) and \( \Lambda = \eta(\lambda - 1) \)
and rearranging the terms, the maximization problem becomes:

$$\max_x \left\{ (1 - \Lambda)(1 - F(y))^{k-1}V_1 + (1 - \Lambda)(k-1)(1 - F(y))^{k-2}F(y)V_2 - cx + \Lambda(1 - F(y))^{2k-2}V_1 + (\Lambda)(k-1)^2(1 - F(y))^{2k-4}F^2(y)V_2 + 2\Lambda(k-1)(1 - F(y))^{2k-3}F(y)V_2 \right\}.$$ (22)

Using the strict monotonicity of $b$ and symmetry, the first order condition (FOC) is given by:

$$\begin{align*}
&(- (1 - \Lambda)(k-1)(1 - F(y))^{k-2}F'(y)y' - \Lambda(2k - 2)(1 - F(y))^{2k-3}F'(y)y' V_1 1/y \\
&+ (- (1 - \Lambda)(k-1)(1 - F(y))^{k-3}F'(y)y'(1 - (k - 1))F(y) \\
&+ 2\Lambda(k - 1)(1 - F(y))^{2k-5}F'(y)y'(1 - kF(y) - ((k - 1)^2 - 1)F(y)^2) V_2 1/y = 1 . \end{align*}$$ (23)

A contestant with the highest possible type $c = 1$ never wins a prize under the assumption $k > 2$. Thus the optimal effort of this contestant is always 0, providing $y(0) = 1$ as a boundary condition.

Note that the FOC is a differential equation with separated variables, since the left hand side of the equation (33) is a function of $y$ only. Denote

$$H(y) = V_1 \left( (1 - \Lambda)(k-1) \int_y^1 \frac{1}{t} (1 - F(t))^{k-2}F'(t)dt + \Lambda(2k - 2) \int_y^1 \frac{1}{t} (1 - F(t))^{2k-3}F'(t)dt \right) \\
+ V_2 \left( (1 - \Lambda)(k-1) \int_y^1 \frac{1}{t} (1 - F(t))^{k-3}(1 - (k - 1))F(t)F'(t)dt \right) \\
+ 2\Lambda(k - 1) \int_y^1 \frac{1}{t} (1 - F(t))^{2k-5}F'(t)(1 - kF(t) - ((k - 1)^2 - 1)F(t)^2)dt) .$$

The solution to the differential equation (33) with the boundary condition $y(0) = 1$ becomes:
\[ \int_{x}^{0} dt = -H(y). \] (24)

Equation (28) gives \( x = H(y) = H(b^{-1}(x)) \) implying \( b = H \). In other words, the effort function of each player is given by \( b(c) = A(c)V_1 + B(c)V_2 \), where

\[
A(c) = (1 - \Lambda) \int_{c}^{1} \frac{1}{a} (k - 1)(1 - F(a))^{k-2} F'(a) da \\
+ \Lambda \int_{c}^{1} \frac{1}{a} (2k - 2)(1 - F(a))^{2k-3} F'(a) da
\]

and

\[
B(c) = (1 - \Lambda) \int_{c}^{1} \frac{1}{a} (k - 1)(1 - F(a))^{k-3} (-1 + (k - 1)F(a)) F'(a) da \\
+ \Lambda \int_{c}^{1} \frac{1}{a} (2k - 2)(1 - F(a))^{2k-5} (-1 + kF(a) + ((k - 2)^2 - 1)F(a)^2) F'(a) da.
\]

Note that the terms multiplied by \( \Lambda \) and \((1 - \Lambda)\) in \( A(c) \) correspond to \(-F_1'(a)\) and \(-F_2'(a)\), and in \( B(c) \) correspond to \(-F_1'(a)\) and \(-((F_2'(a))^2 + (2F_1(a)F_2(a)))'\) respectively, yielding

\[
A(c) = (1 - \Lambda) \int_{c}^{1} \frac{1}{a} F_1'(a) da + \Lambda \int_{c}^{1} \frac{1}{a} (F_1'(a))^2 da
\] (25)

and

\[
B(c) = (1 - \Lambda) \int_{c}^{1} \frac{1}{a} F_2'(a) da + \Lambda \int_{c}^{1} \frac{1}{a} ((F_2'(a))^2 + (2F_1(a)F_2(a)))' da.
\] (26)

It remains to show that the equilibrium effort function \( b(c) \) is differentiable and strictly decreasing. The former one is obvious. To show that the effort function is
strictly decreasing, consider the derivatives of $A(c)$ and $B(c)$:

\[ A'(c) = (1 - \Lambda) - \frac{1}{c}(k - 1)(1 - F(c))^{k-2}F'(c) \]
\[ - \Lambda \frac{1}{c}(2k - 2)(1 - F(c))^{2k-3}F'(c) < 0 \]

and

\[ B'(c) = (1 - \Lambda) - \frac{1}{c}(k - 1)(1 - F(c))^{k-3}(-1 + (k - 1)F(c))F'(c) \]
\[ + \Lambda - \frac{1}{c}(2k - 2)(1 - F(c))^{2k-5}(-1 + kF(c) + ((k - 2)^2 - 1)F(c)^2)F'(c). \]

The derivative of the effort function $b(c)$ becomes:

\[ b'(c) = A'(c)V_1 + B'(c)V_2 \]
\[ \leq V_2 (A'(c) + B'(c)) \]
\[ < 0 \]

since $V_2 \leq V_1$ and $B'(c)$ is smaller than $A'(c)$ in magnitude. Thus $b(c)$ is strictly decreasing.

Now suppose that contestants are sufficiently loss-averse, that is $\Lambda > 1$. In this case equation (31) implies that a non-negative expected pay-off from participating in the contest results for a contestant with type $c$ only if

\[ \frac{(F_1(c))^2V_1 + (F_2(c))^2V_2 + 2F_1(c)F_2(c)V_2}{F_1(c)V_1 + F_2(c)V_2} > 1 - \frac{1}{\Lambda}. \]  \hspace{1cm} (27)

By Lemma[1] there exists a critical type, $\hat{c}$, such that for all types $c < \hat{c}$ equation [27] is satisfied while for all types $c > \hat{c}$ it is violated. In order to secure a nonnegative pay-off all contestants with $c > \hat{c}$ exert 0 effort in equilibrium. Note that $\hat{c} = 1$.
whenever $\Lambda \leq 1$.

The maximization problem of the agents remains the same, however the boundary condition becomes $y(0) = \tilde{c}$ when $\Lambda > 1$. Denote

$$
\tilde{H}(y) = \begin{aligned}
V_1 \left( (1 - \Lambda)(k - 1) \int_y^{\tilde{c}} \frac{1}{t}(1 - F(t))^{k-2}F'(t)dt + \Lambda(2k - 2) \int_y^{\tilde{c}} \frac{1}{t}(1 - F(t))^{2k-3}F'(t)dt \right) \\
+ V_2 \left( (1 - \Lambda)(k - 1) \int_y^{\tilde{c}} \frac{1}{t}(1 - F(t))^{k-3}(1 - (k - 1))F(t)F'(t)dt \\
+ 2\Lambda(k - 1) \int_y^{\tilde{c}} \frac{1}{t}(1 - F(t))^{2k-5}F'(t)(1 - kF(t) - ((k - 1)^2 - 1)F(t)^2)dt \right).
\end{aligned}
$$

The solution to the differential equation (33) with the new boundary condition becomes:

$$
\int_x^0 dt = -\tilde{H}(y). \quad (28)
$$

Using the same arguments as in the case of $\Lambda \leq 1$, the effort function of each contestant with type $c \leq \tilde{c}$ is given by $b(c) = A(c)V_1 + B(c)V_2$, where

$$
A(c) = (1 - \Lambda) \int_c^{\tilde{c}} \frac{1}{a}(k - 1)(1 - F(a))^{k-2}F'(a)da \\
+ \Lambda \int_c^{\tilde{c}} \frac{1}{a}(2k - 2)(1 - F(a))^{2k-3}F'(a)da
$$

and

$$
B(c) = (1 - \Lambda) \int_c^{\tilde{c}} \frac{1}{a}(k - 1)(1 - F(a))^{k-3}(-1 + (k - 1)F(a)) F'(a)da \\
+ \Lambda \int_c^{\tilde{c}} \frac{1}{a}(2k - 2)(1 - F(a))^{2k-5}(-1 + kF(a) + ((k - 2)^2 - 1)F(a)^2) F'(a)da.
$$

Substituting $F_1(c)$ and $F_2(c)$, the weights of the first and the second prizes be-
come:

\[ A(c) = (1 - \Lambda) \int_{c}^{\hat{c}} -\frac{1}{a} F_1'(a) da + \Lambda \int_{c}^{\hat{c}} -\frac{1}{a} (F_2' (a))' da \]  \hspace{1cm} (29)

and

\[ B(c) = (1 - \Lambda) \int_{c}^{\hat{c}} -\frac{1}{a} F_2'(a) da + \Lambda \int_{c}^{\hat{c}} -\frac{1}{a} ( (F_2^2 (a))' + (2 F_1(a)F_2(a))') da. \]  \hspace{1cm} (30)

Note that the weights of the first and the second prize are the same for any value of \( \Lambda \), with the critical type being the least able contestant \( \hat{c} = 1 \) whenever \( \Lambda \leq 1 \).

The optimal effort function is differentiable and strictly decreasing when \( \hat{c} < 1 \), similar to the case of \( \hat{c} = 1 \).

**Proof of Proposition 3**

The equilibrium effort function in the case of convex or concave cost-of-effort is derived in a similar to the case of linear cost-of-effort. As in the case of linear cost-of-effort, I will derive the optimal effort function first for the case when there is full participation in the contests (when \( \Lambda \leq 1 \)) and then for the case when some contestants drop out the contest (when \( \Lambda > 1 \)).

Assume that all contestants except \( i \) exert effort according to the function \( b \) which is strictly monotonic and differentiable. Suppose that each contestant participates in the contest, that is \( \Lambda \leq 1 \). The maximization problem of the contestant \( i \) with convex or concave cost-of-effort \( \gamma(x) \) is:

\[
\max x \quad \{p_1 \{V_1 + \eta(p_2(V_1 - V_2) + (1 - p_1 - p_2)V_1) - c\gamma(x)\} \\
+ p_2 \{V_2 + \eta(p_1\lambda(V_2 - V_1) + (1 - p_1 - p_2)V_2) - c\gamma(x)\} \} \\
+ (1 - p_1 - p_2)\{\eta(p_1\lambda(-V_1) + p_2\lambda(-V_2)) - c\gamma(x)\} \}. \]  \hspace{1cm} (31)

where the probabilities of winning the first and the second prize, \( p_1 \) and \( p_2 \), are
defined as in equation (21).

Denote the inverse effort function \( b^{-1} \) by \( y \). Substituting \( b^{-1} \) and \( \Lambda = \eta(\lambda - 1) \) and rearranging the terms, the maximization problem becomes:

\[
\max_x \left\{ (1 - \Lambda)(1 - F(y))^{k-1}V_1 + (1 - \Lambda)(k - 1)(1 - F(y))^{k-2}F(y)V_2 \\
- c\gamma(x) + \Lambda(1 - F(y))^{2k-2}V_1 + (\Lambda)(k - 1)^2(1 - F(y))^{2k-4}F^2(y)V_2 \\
+ 2\Lambda(k - 1)(1 - F(y))^{2k-3}F(y)V_2 \right\}.
\]

(32)

Using the strict monotonicity of \( b \) and symmetry, the first order condition (FOC) is given by:

\[
(- (1 - \Lambda)(k - 1)(1 - F(y))^{k-2}F'(y)y' - \Lambda(2k - 2)(1 - F(y))^{2k-3}F''(y)y') V_1 \frac{1}{y} \\
+ (- (1 - \Lambda)(k - 1)(1 - F(y))^{k-3}F'(y)y'(1 - (k - 1))F(y) \\
+ 2\Lambda(k - 1)(1 - F(y))^{2k-5}F'(y)y'(1 - kF(y) - ((k - 1)^2 - 1)F(y)^2) V_2 \frac{1}{y} = \gamma(33)
\]

Using the boundary condition \( y(0) = 1 \), the solution to this differential equation is given by \( \gamma(x) = H(y) \), where \( H(y) \) is given by equation [24]. Thus \( x = \gamma^{-1}(H(y)) \) implying that \( b = \gamma^{-1}(H) \). The effort function of each contestant is given by \( b(c) = \gamma^{-1} (A(c)V_1 + B(c)V_2) \), where \( A(c) \) and \( B(c) \) are given by equation [9] and [10] with \( \tilde{c} = 1 \) respectively.

Now suppose that contestants are sufficiently loss-averse, that is \( \Lambda > 1 \). In this case by Lemma [1] there exists a critical type, \( \tilde{c} \), such that for all types \( c < \tilde{c} \) equation (27) is satisfied while for all types \( c > \tilde{c} \) it is violated. Recall that contestants with \( c > \tilde{c} \) exert 0 effort in equilibrium. Note that \( \tilde{c} = 1 \) whenever \( \Lambda \leq 1 \).

The maximization problem of the agents remains the same, however the boundary condition becomes \( y(0) = \tilde{c} \) when \( \Lambda > 1 \). The solution to the differential
equation (33) with the new boundary condition becomes \( \gamma(x) = \tilde{H}(y) \), where \( \tilde{H}(c) \) is given by equation (28). The effort function of each contestant is then given by \( b(c) = \gamma^{-1}(A(c)V_1 + B(c)V_2) \), where \( A(c) \) and \( B(c) \) are given by equations (9) and (10) respectively.

It remains to show that the equilibrium effort function \( b(c) \) is differentiable and strictly decreasing. The former one is obvious. To show that the effort function is strictly decreasing, consider the derivative of the effort function, \( b'(c) \):

\[
b'(c) = \gamma^{-1}(A(c)V_1 + B(c)V_2) (A'(c)V_1 + B'(c)V_2) < 0
\]

Using the proof of Proposition 1 and the fact that \( \gamma^{-1} > 0 \), one concludes that \( b(c) \) is strictly decreasing.

B. Optimal Allocation of Prizes

Proof of Proposition 2. Assume that there are two prizes \( V_1 \geq V_2 \geq 0 \) to be awarded and \( k > 2 \) contestants. Assume that contestants have linear cost-of-effort functions. By Proposition 1 the average effort of each contestant is given by:

\[
\int_{m}^{\tilde{c}} b(c) F'(c) dc = \int_{m}^{\tilde{c}} (1 - \alpha) A(c) + \alpha B(c) F'(c) dc.
\] (34)

where \( A(c) \) and \( B(c) \) are given by equations (9) and (10). Note that \( \tilde{c} = 1 \) whenever \( \Lambda \leq 1 \). The designer’s problem becomes:

\[
\max_{0 \leq \alpha \leq 1/2} k \int_{m}^{\tilde{c}} (A(c) + \alpha(B(c) - A(c))) F'(c) dc.
\] (35)

Equivalently

\[
\max_{0 \leq \alpha \leq 1/2} \alpha \int_{m}^{\tilde{c}} (B(c) - A(c)) F'(c) dc.
\] (36)
It is optimal to award a single first prize if and only if the integral in equation (36) is negative. Otherwise the optimal prize structure consists of two equal prizes, due to the linearity of the program.

**Proof of Proposition 4.** Assume that there are two prizes \( V_1 \geq V_2 \geq 0 \) to be awarded and \( k > 2 \) contestants with either convex or concave cost-of-effort functions. By Proposition 3 the average effort of each contestant is given by:

\[
\int_{\tilde{c}} b(c) F'(c) dc = \int_{\tilde{c}} \gamma^{-1} ((1 - \alpha) A(c) + \alpha B(c)) F'(c) dc. \tag{37}
\]

where \( A(c) \) and \( B(c) \) are given by equations (9) and (10). Note that \( \tilde{c} = 1 \) whenever \( \Lambda \leq 1 \). The designer’s revenue is given by:

\[
R(\alpha) = k \int_{\tilde{c}} \gamma^{-1} ((1 - \alpha) A(c) + \alpha B(c)) F'(c) dc. \tag{38}
\]

The designer’s problem becomes:

\[
\max_{0 \leq \alpha \leq 1/2} k \int_{\tilde{c}} \gamma^{-1} ((1 - \alpha) A(c) + \alpha B(c)) F'(c) dc. \tag{39}
\]

If condition in equation (19) is not satisfied, that is \( R'(0) < 0 \), then the integral in equation (39) is maximized at \( \alpha = 0 \). If, however, condition in equation (19) is satisfied, then \( R(\alpha) \) can not have a maximum at \( \alpha = 0 \). It has a maximum at \( \alpha^* \) with \( R'(\alpha^*) = 0 \).

**Proof of Corollary 1.** The revenue of the contest designer is given by:

\[
R(\alpha) = k \int_{m}^{\tilde{c}} \gamma^{-1}(A(c) + \alpha(B(c) - A(c))) F'(c) dc.
\]

Taking the second derivative of the revenue function with respect to \( \alpha \) we get:
\[ R''(\alpha) = k \int_m^1 \gamma^{-1''}(A(c) + \alpha(B(c) - A(c)))(B(c) - A(c))^2 F'(c) dc \]

Since the cost-of-effort function is concave, \( \gamma^{-1} \) is convex so that \( \gamma^{-1''} > 0 \). Combining the two, \( R''(\alpha) > 0 \) implying that \( R(\alpha) \) is convex in \( \alpha \). Therefore, the maximum of the revenue function is either at corner values, either \( \alpha = 0 \) or \( \alpha = 0.5 \). Note that \( (B(c) - A(c))^2 > 0 \). Combining the two, \( R''(\alpha) > 0 \) implying that \( R(\alpha) \) is convex in \( \alpha \). Therefore, the maximum of the revenue function is either at corner values, either \( \alpha = 0 \) or \( \alpha = 0.5 \). Note that \( (B(c) - A(c))^2 \) can not be zero, since there is always a positive mass of contestants exerting positive effort. Put differently, there exists a \( \epsilon > 0 \) such that contestants with abilities in \( (m, m + \epsilon) \) exert positive effort.

C. The Symmetric Equilibrium with \( p \) Prizes

Assume that there are \( 2 < p \leq k \) prizes to be awarded with \( V_1 \geq V_2 \geq \cdots \geq V_p \) and \( k > p \) contestants. Assume that the cost-of-effort of contestants is given by \( c\gamma(x) \), where \( \gamma \) is allowed to be linear, convex or concave. \( F_i(a) \) denotes the probability that contestant \( i \) with type \( a \) meets \( k - 1 \) competitors such that \( s - 1 \) of these competitors have lower types than \( i \) and remaining \( k - s \) competitors have higher types than \( i \). \( F_s(a) \) is then given by:

\[ F_s(a) = \binom{k-1}{s-1} (1 - F(a))^{k-2} (F(a))^{s-1} \]

The expected utility of contestant \( i \) with cost parameter \( c \) is given by:

\[
EU = \sum_{p=1}^{P} F_p V_p + \eta \left( \sum_{i>p} F_p F_i (V_p - V_i) + \sum_{i<p} F_p F_i \lambda(V_p - V_i) \right) - F_p c \gamma(x) \\
+ \eta \sum_{p=1}^{P} (1 - \sum_{i=1}^{P} F_i) F_p \lambda(0 - V_p) \\
+ (1 - \sum_{i=1}^{P} F_i) c \gamma(x).
\]
Substituting $\Lambda = \eta(\lambda - 1)$ and rearranging the terms, the expected utility of contestant $i$ becomes:

$$EU = \sum_{p=1}^{P} F_p V_p - c\gamma(x)$$

$$- \Lambda \left\{ \sum_{i<p}^P F_p F_i(V_p - V_i) + \sum_{i=1}^{P} F_p(1 - \sum_{i=1}^{P}) V_p \right\}.$$

Rearranging the terms, one gets:

$$EU = (1 - \Lambda) \sum_{p=1}^{P} F_p V_p + \Lambda \left\{ \sum_{p=1}^{P} F_p^2 V_p + \sum_{i<p,p=2}^{P} 2V_p F_p \sum_{i} F_1 \right\} - c\gamma(x).$$

The maximization problem of contestant $i$ reads:

$$\max_x (1 - \Lambda) \sum_{p=1}^{P} F_p V_p + \Lambda \left\{ \sum_{p=1}^{P} F_p^2 V_p + \sum_{i<p,p=2}^{P} 2V_p F_p \sum_{i} F_1 \right\} - c\gamma(x).$$

First order condition becomes:

$$\sum_{p=1}^{P} V_p \left\{ (1 - \Lambda) F'_p + \Lambda \left( (F_p^2)' + \sum_{i<p} (2F_p F_i)' \right) \right\} = c\gamma^{-1'}(x)$$

The equilibrium effort function of contestant $i$ with cost parameter $c$ whose loss-aversion degree is smaller than 1, that is $\Lambda \leq 1$, becomes:
\[ b(c) = \gamma^{-1} \left( \sum_s^p V_s \left\{ (1 - \Lambda) \int_c^1 \frac{1}{a} F_s(a)' \, da + \Lambda \int_c^1 \frac{1}{a} \left( (F_s(a)^2)' + \sum_{i=1}^{s-1} (2F_i(a)F_s(a))' \right) \, da \right\} \right) \]  

(40)

Whenever \( \Lambda > 1 \), analogous to Lemma \( \text{[L]} \) there exist a critical type \( \tilde{c} \) satisfying the following equation

\[
\frac{\sum_{s=1}^p (F_s(\tilde{c}))^2 V_s + \sum_{s=2, j<s}^p 2V_s F_s(\tilde{c}) F_j(\tilde{c})}{\sum_{s=1}^p F_s(\tilde{c}) V_s} = 1 - \frac{1}{\Lambda}.
\]

such that any contestant with \( c \geq \tilde{c} \) exerts zero effort in equilibrium, while contestants with \( c < \tilde{c} \) exert effort in equilibrium according to equation (40).

**D. Allocation of \( p \) Prizes for Linear Costs**

Assume that there are \( 2 < p \leq k \) prizes to be awarded with \( V_1 \geq V_2 \geq ... \geq V_{p-1} \geq V_p \geq 0 \) and \( k > 2 \) contestants. \( F_s(a) \) denotes the probability that a contestant with type \( a \) wins the \( s \)-th prize, given by

\[
F_s(a) = \binom{k-1}{s-1} (1 - F(a))^{k-2}(F(a))^{s-1}.
\]

If \( \Lambda > 1 \), then there exists a critical type \( \tilde{c} \) such that in equilibrium each contestant with \( c \geq \tilde{c} \) exerts 0 effort while each contestant with \( c < \tilde{c} \) exerts effort according to

\[
b(c) = \sum_s^p V_s \left\{ (1 - \Lambda) \int_c^{\tilde{c}} \frac{1}{a} F_s(a)' \, da + \Lambda \int_c^{\tilde{c}} \frac{1}{a} \left( (F_s(a)^2)' + \sum_{i=1}^{s-1} (2F_i(a)F_s(a))' \right) \, da \right\}.
\]
If $\Lambda \leq 1$, each contestant exerts effort according to the above equation with $\tilde{c} = 1$.

Denote the coefficient of $V_s$ by $A_s$:

$$A_s = \left\{ (1 - \Lambda) \int_c^{\bar{c}} \frac{1}{a} F_s(a) \, da \
+ \Lambda \int_c^{\bar{c}} \frac{1}{a} \left( (F_s(a)^2)' + \sum_{i=1}^{s-1} (2F_i(a)F_s(a))' \right) \, da \right\}.$$  

Substituting this into the bidding function we get:

$$b(c) = \sum_{s=1}^{p} V_s A_s(c)$$

$$= \left( 1 - \sum_{i=1}^{p-1} V_{i+1} \right) A_1(c) + \sum_{i=2}^{p} V_i A_i$$

$$= A_1 + \sum_{i=2}^{p} V_i (A_i(c) - A_1(c))$$

The designer’s problem becomes:

$$\max_{0 \leq V_i \leq \frac{1}{k}} k \int_m^{\bar{c}} \left\{ A_1(c) + \sum_{i=2}^{p} V_i (A_i(c) - A_1(c)) \right\} F'(c) \, dc$$

subject to the following $p - 1$ conditions:

$$1 - \sum_{i=1}^{p} V_i \geq V_2$$
$$V_2 \geq V_3$$

$$\vdots$$
$$V_{p-1} \geq V_p$$

Since $A_1$ does not have a coefficient of type $V_i$, deleting $A_1$ would not harm. Since the summation is finite, it is allowed to interchange the integral and the summation.
signs. Then the maximization problem reads:

$$\max_{0 \leq V_i \leq \frac{1}{2}} \sum_{i=2}^{p} \left\{ V_i \int_{m}^{e} (A_i(c) - A_1(c)) F'(c) dc \right\}$$

subject to equation (42). It is optimal to award a single first prize if and only if each summand in the maximization problem is zero, that is

$$\int_{m}^{e} (A_i(c) - A_1(c)) F'(c) dc < 0.$$  

for each $i \in \{2, \ldots, p\}$. Otherwise, it is optimal to award equal prizes only, due to the linearity of the program. That is the constraints in equation (42) will all bind. To see this, suppose to the contrary that there is an interior solution. WLOG assume that the interior solution is $(\sigma, \varsigma, \tau, 0, \ldots, 0) \in [0, 1]^p$, where $\sigma > \varsigma > \tau$ and $\sigma + \varsigma + \tau = 1$.

For the sake of easiness denote $G_i := \int_{m}^{e} (A_i(c) - A_1(c)) F'(c) dc > 0$. Since this allocation is optimal, and $\sigma, \varsigma, \tau$ are all positive it means that $G_4$ is positive (otherwise it would be optimal to transfer the weight $\tau$ to $\sigma$ and $\varsigma$). Since $\tau > 0$, $G_2$ should be greater than both $G_3$ and $G_4$ (otherwise it would be optimal to transfer the weight $\tau$ to $\sigma$ and $\varsigma$). But then $\tau$ should take the biggest value it could take, which is in this case $\frac{1}{3}$ (otherwise $(\sigma, \varsigma, \tau, 0, \ldots, 0)$ would not be optimal). Applying the same reasoning to both $\sigma$ and $\varsigma$, we conclude that $\sigma = \varsigma = \tau = \frac{1}{3}$. In order to obtain the optimal prize allocation one needs to evaluate the objective function only on the boundary values, namely on the set $\{(1,0,\ldots,0),(\frac{1}{2},\frac{1}{2},0,\ldots,0),\ldots,(\frac{1}{p},\frac{1}{p},\ldots,\frac{1}{p})\}$ and take the allocation which gives the maximum value. It is optimal to award $2 \leq r \leq p$ equal prices if and only if

$$r = \arg \max_{j \in 2,\ldots,p} \frac{1}{j} \sum_{i=2}^{j} \left\{ \int_{m}^{e} (A_i(c) - A_1(c)) F'(c) dc \right\}.$$
E. Allocation of $p$ Prizes for Convex or Concave Costs

Assume that there are $2 < p \leq k$ prizes to be awarded with $V_1 \geq V_2 \geq \ldots \geq V_{p-1} \geq V_p \geq 0$ and $k > 2$ contestants. As before, $F_s(a)$ denotes the probability that a contestant with type $a$ wins the $s$-th prize, given by

$$F_s(a) = \left(\frac{k-1}{s-1}\right) (1 - F(a))^{k-2} (F(a))^{s-1}.$$ 

Whenever $\Lambda > 1$, there exists a critical type $\tilde{c}$ satisfying the following equation

$$\frac{\sum_{s=1}^{p}(F_s(\tilde{c}))^2V_s + \sum_{s=2,i<s}^{p} 2V_i F_s(\tilde{c}) F_i(\tilde{c})}{\sum_{s=1}^{p} F_s(\tilde{c})V_s} = 1 - \frac{1}{\Lambda},$$

such that any contestant with $c \geq \tilde{c}$ exerts zero effort in equilibrium, while contestants with $c < \tilde{c}$ exert effort in equilibrium according to

$$b(c) = \gamma^{-1} \left( \sum_s^p V_s \left\{ (1 - \Lambda) \int_c^\tilde{c} - \frac{1}{a} F_s(a)' da \\ + \Lambda \int_c^\tilde{c} - \frac{1}{a} \left( (F_s(a)^2)' + \sum_{i=1}^{s-1} (2F_i(a)F_s(a))' \right) da \right\} \right)$$

Note that $\tilde{c} = 1$ whenever $\Lambda \leq 1$. Denote the coefficient of $V_s$ by $A_s$:

$$A_s = \left\{ (1 - \Lambda) \int_c^\tilde{c} - \frac{1}{a} F_s(a)' da \\ + \Lambda \int_c^\tilde{c} - \frac{1}{a} \left( (F_s(a)^2)' + \sum_{i=1}^{s-1} (2F_i(a)F_s(a))' \right) da \right\}.$$
Substituting this into the optimal effort function we get:

\[ b(c) = \gamma^{-1} \left( \sum_{s=1}^{p} V_s A_s(c) \right) \]

\[ = \gamma^{-1} \left( 1 - \sum_{i=1}^{p-1} V_{i+1} \right) A_1(c) + \sum_{i=2}^{p} V_i A_i \]

\[ = \gamma^{-1} \left( A_1 + \sum_{i=2}^{p} V_i (A_i(c) - A_1(c)) \right) \]

The designer's problem becomes:

\[ \max_{0 \leq V_i \leq \frac{1}{p}} k \int_{m}^{c} \gamma^{-1} \left( A_1(c) + \sum_{i=2}^{p} V_i (A_i(c) - A_1(c)) \right) F'(c) dc \]

subject to the following \( p - 1 \) conditions:

\[ 1 - \sum_{i=1}^{p} V_i \geq V_2 \quad (42) \]
\[ V_2 \geq V_3 \]
\[ \vdots \]
\[ V_{p-1} \geq V_p \]

So that the designer's revenue is:

\[ R(V_2, \cdots, V_p) = k \int_{m}^{c} \gamma^{-1} \left( A_1(c) + \sum_{i=2}^{p} V_i (A_i(c) - A_1(c)) \right) F'(c) dc \]

For the designer it is optimal to allocate the total prize sum into a single first one only if the partial derivatives \( \frac{\partial R}{\partial V_h}(V_2, \cdots, V_p) \leq 0 \) for each \( h \in \{2, \cdots, P\} \). Thus, a sufficient for the optimality of multiple prizes is given by:

\[ \int_{m}^{c} \gamma^{-1} \left( A_1 + \sum_{i=2, i \neq h}^{p} V_i (A_i - A_1) \right) (A_h - A_1) dF'(c) dc > 0, \]

for each \( h \in \{2, \cdots, P\} \).
References


